

Normal Track

The Expanded Train and a Bomb

Example

A train with proper lengths rL (with $0 < r \leq 1$) moves at speed v towards a tunnel with proper length L . A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, this sensor tells the bomb to disarm itself.

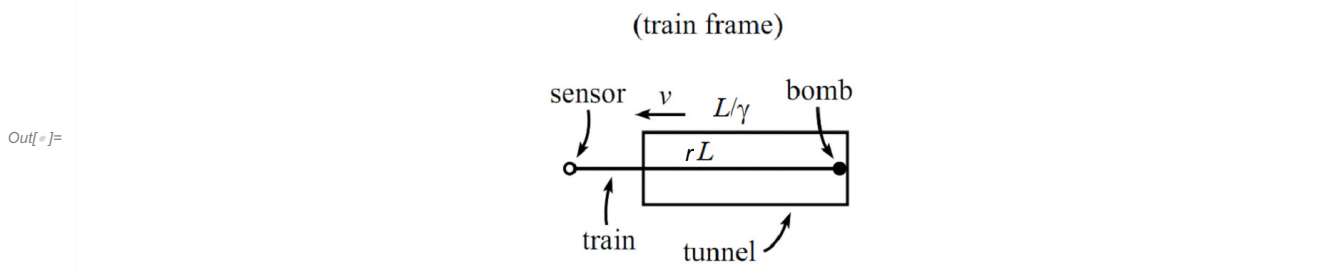
- In the train's frame, what is the largest value of r in terms of v for which it is possible for the bomb not to explode?

Hint: Set $t = 0$ when the rear of the train enters the tunnel and solve for the time when the deactivation signal reaches the front of the train and the time when the front of the train emerges from the tunnel.

- Redo this analysis in the tunnel's frame. Do you get the same answer?

Solution

In the train's frame, the length contracted tunnel moves towards the stick. If $r \geq \frac{1}{\gamma}$, the bomb will clearly explode (in fact, $r = 1$ is the traditional Ladder in a Barn question).



Assuming $r < \frac{1}{\gamma}$, we set $t = 0$ as the time when the rear of the train (with the sensor) enters the tunnel. At this point, the sensor sends a light signal towards the front of the train, and it is a race between that photon and the front of the train emerging from the tunnel exit.

The time it takes for the photon to reach the front of the (stationary) train is $\frac{rL}{c}$, while the time it takes the back of the length contracted tunnel to reach the front of the train (with the bomb) equals $\frac{L-rL}{v}$. Setting these two quantities equal yields

$$\frac{rL}{c} = \frac{L-rL}{v} \tag{1}$$

$$r = \frac{1}{1 + \frac{v}{c} \gamma} \tag{2}$$

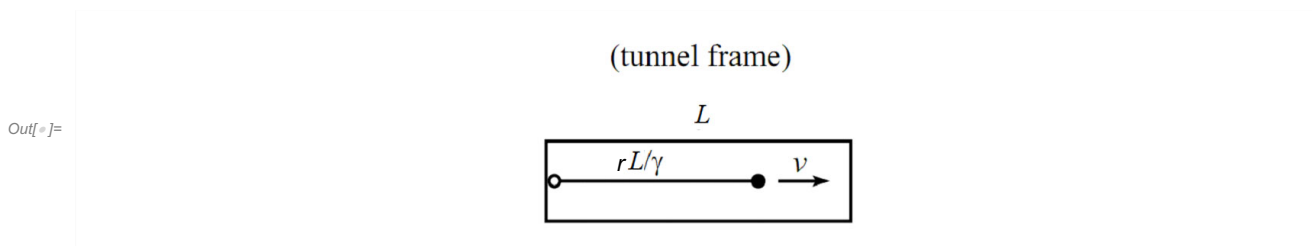
which, as expected, is less than $\frac{1}{\gamma}$!

$$\text{In}[*]:= \text{Simplify}[\text{Solve}[\frac{rL}{c} == \frac{L - rL}{v}, r] /. \gamma \rightarrow \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}]$$

$$\text{Out}[*]:= \left\{ \left\{ r \rightarrow \frac{c \sqrt{1 - \frac{v^2}{c^2}}}{c + v} \right\} \right\}$$

Redoing this analysis from the tunnel's frame, we now see the length contracted stick entering the tunnel as shown below. Setting $t = 0$ as before, a photon fired from the back of the train must reach the front of the train (a net distance $\frac{rL}{\gamma}$) while moving at a speed $c - v$ relative to the front of the train (since the train is moving away from

the photon), which will happen in time $\frac{rL}{c-v}$. Meanwhile, the front of the train approaches the rear of the tunnel, and the collision between them will occur in $\frac{L - rL}{v}$.



Setting these two quantities equal to each other yields

$$\frac{rL}{c-v} = \frac{L - rL}{v} \tag{3}$$

$$r = \frac{1}{1 + \frac{v}{c}} \frac{1}{\gamma} \tag{4}$$

$$\text{In}[*]:= \text{Simplify}[\text{Solve}[\frac{rL}{c - v} == \frac{L - rL}{v}, r] /. \gamma \rightarrow \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}]$$

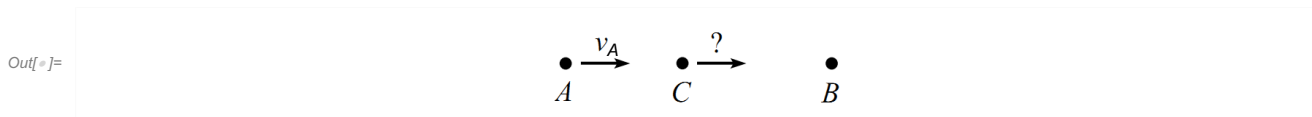
$$\text{Out}[*]:= \left\{ \left\{ r \rightarrow \frac{c \sqrt{1 - \frac{v^2}{c^2}}}{c + v} \right\} \right\}$$

Note that the two values of r had to be the same in both frames, because the train cannot explode in one reference frame but not in another (whether the train explodes or not is frame independent). □

More Equal Speeds

Example

A moves at speed v_A , and B is at rest. At what speed v_C must C travel, so that she sees A and B approaching her at the same rate?



Suppose that A and C arrive at B at the same time. In the lab frame (B 's frame), what is the ratio of the distances CB and AC ? (The answer to this is very nice and clean. In such cases, you should think of a simple, intuitive explanation for the result!)

Solution

Denote A 's, B 's, and C 's speeds by v_A , v_B , and v_C , respectively. Let us boost all of the speeds by v_C to the left to go into C 's frame.

Let's begin with the two easy ones. Boosting C 's speed to the left will result in no velocity, by construction. Next, since B is at rest, boosting its speed by v_C to the left yields the speed $-v_C$ pointing to the right (or a speed v_C pointing to the left). Finally, boosting A 's speed to the left by v_C yields the velocity $\frac{v_A - v_C}{1 - \frac{v_A v_C}{c^2}}$ pointing to the right. In

order for A and B to approach C at the same speed from both directions, we must have

$$\frac{v_A - v_C}{1 - \frac{v_A v_C}{c^2}} = v_C \quad (5)$$

Solving this using *Mathematica*,

Simplify[Solve[$\frac{v_A - v_C}{1 - \frac{v_A v_C}{c^2}} = v_C, v_C], c > 0]$

$$\left\{ \left\{ v_C \rightarrow \frac{c \left(c - \sqrt{c^2 - v_A^2} \right)}{v_A} \right\}, \left\{ v_C \rightarrow \frac{c \left(c + \sqrt{c^2 - v_A^2} \right)}{v_A} \right\} \right\}$$

Of the two solutions $v_C = \frac{c^2}{v_A} \left(1 \pm \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2} \right)$, only the minus sign solution is physical (i.e. less than c), and hence the speed at which C must travel is $v_C = \frac{c^2}{v_A} \left(1 - \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2} \right)$.

If A and C arrive at B at the same time (note that the two events - A arriving at B and C arriving at B - occur at the same time *and* place; therefore they occur simultaneously in *all* frames), then the ratio of the distances will equal

$$\begin{aligned} \frac{CB}{AC} &= \frac{v_C - v_B}{v_A - v_C} \\ &= \frac{\frac{c^2}{v_A} \left(1 - \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2} \right)}{v_A - \frac{c^2}{v_A} \left(1 - \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2} \right)} \\ &= \frac{1 - \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2}}{\left(\frac{v_A}{c} \right)^2 - \left(1 - \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2} \right)} \quad (\text{see comment below}) \\ &= \frac{1}{\left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2}} \\ &= \gamma_A \end{aligned} \quad (6)$$

where in the third step we divided by $\left(1 - \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2} \right)$ and used the relation

$$\left(1 - \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2} \right) \left(1 + \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2} \right) = 1 - \left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\} = \left(\frac{v_A}{c} \right)^2 \quad (7)$$

while in the last step we defined

$$\gamma_A = \frac{1}{\left\{ 1 - \left(\frac{v_A}{c} \right)^2 \right\}^{1/2}} \quad (8)$$

to be A 's γ factor in B 's frame. This implies that C is γ_A as far from B as she is from A . Note that for non-relativistic speeds $v \ll c$, $\gamma_A \approx 1$ and $v_C = \frac{v_A}{2}$ so that C is midway between A and B .

You may (or at least should) be wondering why in the world $\frac{CB}{AC} = \gamma_A$ is such a simple relation. **In physics, getting such clean results demands a correspondingly simple explanation.** Simple answers imply that if we had

considered the problem from a different perspective, we should have easily been able to deduce that $\frac{CB}{AC} = \gamma_A$.

Here is one intuitive reason why the value of $\frac{CB}{AC}$ must come out to be the clean result γ_A . Imagine that in C 's frame, A and B are carrying identical jousting sticks as they run toward C ; by the problem setup it is clear that the tips of both sticks will hit C simultaneously in this frame. Because those two events occur simultaneously at the same point in C 's frame, they occur simultaneously in all frames...including B 's frame! But in B 's frame, B 's stick is uncontracted, while A 's stick is length-contracted by a factor γ_A . So when the tips of the two sticks touch C simultaneously, this forces A to be closer to C than B is by a factor γ_A , as desired. \square

The Triplet Paradox

Example

Consider the following variation of the twin paradox. A , B , and C each have a clock. In A 's reference frame, B flies past A with speed v to the right. When B passes A , they both set their clocks to zero. Also, in A 's reference frame, C starts far to the right and moves to the left with speed v . When B and C pass each other, C sets his clock to read the same as B 's. Finally, when C passes A , they compare the readings on their clocks. At this event, let A 's clock read T_A , and let C 's clock read T_C . Define $\gamma = \frac{1}{(1-\frac{v^2}{c^2})^{1/2}}$.

- (a) Working in A 's frame, show that $T_C = T_A/\gamma$
- (b) Working in B 's frame, show again that $T_C = T_A/\gamma$
- (c) Working in C 's frame, show again that $T_C = T_A/\gamma$

Solution

Part (a): Let the starting distance between A and C at time $t = 0$ be $2d$. In A 's reference frame, B and C will meet each other a distance d away from clock A , with both of these clocks moving at speed v . B 's clock will be running slow by a factor of γ , so it will be showing a time $\frac{d}{v\gamma}$ when B and C meet, and transfer this time over to C .

The time it takes for B and C to meet will equal the time it subsequently takes for A to meet C , since both B and C travel at speed v , and clock C is now retracing B 's path. Since C is moving at speed v , the time $\frac{d}{v\gamma}$ will elapse on clock C between the time it meets clock B and clock A . Therefore, C 's clock will ultimately read $T_C = \frac{2d}{v\gamma}$. Throughout this entire time, A 's clock will read the same amount of time, but without the time dilation factor, $T_A = \frac{2d}{v}$.

Therefore, $T_C = T_A/\gamma$.

Part (b): Now let's look at things in B 's frame, where A moves away from B at velocity v while C chases A at a velocity given by relativistically adding speed v with v . Let B 's clock read t_B when he meets C . Then at this time, B hands off the time t_B to C , and B sees A 's clock read $\frac{t_B}{\gamma}$.

We must now determine how much additional time elapses on A 's clock and C 's clock, by the time they meet.

From the velocity-addition formula, B sees C flying by to the left at speed $v_2 \equiv \frac{2v}{1+\frac{v^2}{c^2}}$. He also sees A fly by to the

left at speed v , but A had a head-start of $v t_B$ in front of C . Therefore, if \tilde{t} is the time between the meeting of B and C and the meeting of A and C (as viewed from B), then $v t_B = (v_2 - v) \tilde{t}$. During this time, A 's time increases by $\frac{\tilde{t}}{\gamma}$

while C 's clock increases by $\frac{\tilde{t}}{\gamma_2}$ where $\gamma_2 = \frac{1}{(1-\frac{v_2^2}{c^2})^{1/2}}$. Thus the total time on clock A is

$$\begin{aligned}
 T_A &= \frac{t_B}{\gamma} + \frac{\tilde{t}}{\gamma} \\
 &= \frac{t_B}{\gamma} + t_B \frac{v}{(v_2 - v)\gamma} \\
 &= \frac{t_B}{\gamma} \left(1 + \frac{1 + \frac{v}{v_2}}{1 - \frac{v}{v_2}} \right) \\
 &= 2\gamma t_B
 \end{aligned}$$

The total time on clock C reads (after some algebra)

$$\begin{aligned}
 T_C &= t_B + \frac{\tilde{t}}{\gamma^2} \\
 &= t_B + \tilde{t} \frac{1 - v^2}{1 + v^2} \\
 &= t_B + t_B \frac{1 + v^2}{1 - v^2} \frac{1 - v^2}{1 + v^2} \\
 &= 2 t_B
 \end{aligned} \tag{10}$$

Therefore, $T_C = T_A / \gamma$.

Part (c): In C 's frame, A and B both approach C , but B does so faster, moving at a velocity of $v_2 \equiv \frac{2v}{1 + \frac{v^2}{c^2}}$ as found in Part b. Denote the starting distance between B and C to be $\tilde{d} \equiv \frac{2d}{\gamma}$ (the length contraction of the distance discussed in Part a (although since we want the ratio of T_A to T_C this length contraction cancels out)). Then B and C meet at time $\frac{\tilde{d}}{v_2}$ (as measured by a stationary observer in C 's reference frame), at which point B 's clock reads $\frac{\tilde{d}}{v_2 \gamma^2}$, which is the time that B passes off to C . It then takes a time $\frac{\tilde{d}}{v} - \frac{\tilde{d}}{v_2}$ for clock A to reach C , which means that C will ultimately read (after some messy algebra, which should simply be done in Mathematica)

$$\begin{aligned}
 T_C &= \frac{\tilde{d}}{v_2 \gamma^2} + \frac{\tilde{d}}{v} - \frac{\tilde{d}}{v_2} \\
 &= \frac{\tilde{d}}{v} \left(\frac{v + v_2 \gamma^2 - v \gamma^2}{v_2 \gamma^2} \right) \\
 &= \frac{\tilde{d}}{v \gamma^2}
 \end{aligned} \tag{11}$$

$$\text{FullSimplify}\left[\frac{d}{v \gamma^2} == \left(\frac{d}{v_2 \gamma^2} + \frac{d}{v} - \frac{d}{v_2}\right) /. \gamma^2 \rightarrow \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} /. \gamma \rightarrow \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} /. v_2 \rightarrow \frac{2v}{1 + \frac{v^2}{c^2}}, \text{Assumptions} \rightarrow 0 < v < c\right]$$

True

The total time it takes for A to reach C (as measured in C 's frame) equals $\frac{\tilde{d}}{v}$, and due to A 's speed v the final time that A reads will be

$$T_A = \frac{\tilde{d}}{v \gamma} \tag{12}$$

Therefore, $T_C = T_A / \gamma$. \square

Hard Track

Deriving Length Contraction

This problem gives a derivation of length contraction that does not rely on time dilation!

Example

Assume that the rule for (longitudinal) length contraction is: "If a stick with proper length L moves at speed v with respect to you, then its length in your frame is $a_v L$." Our goal will be to find a_v . A critical point here is that the

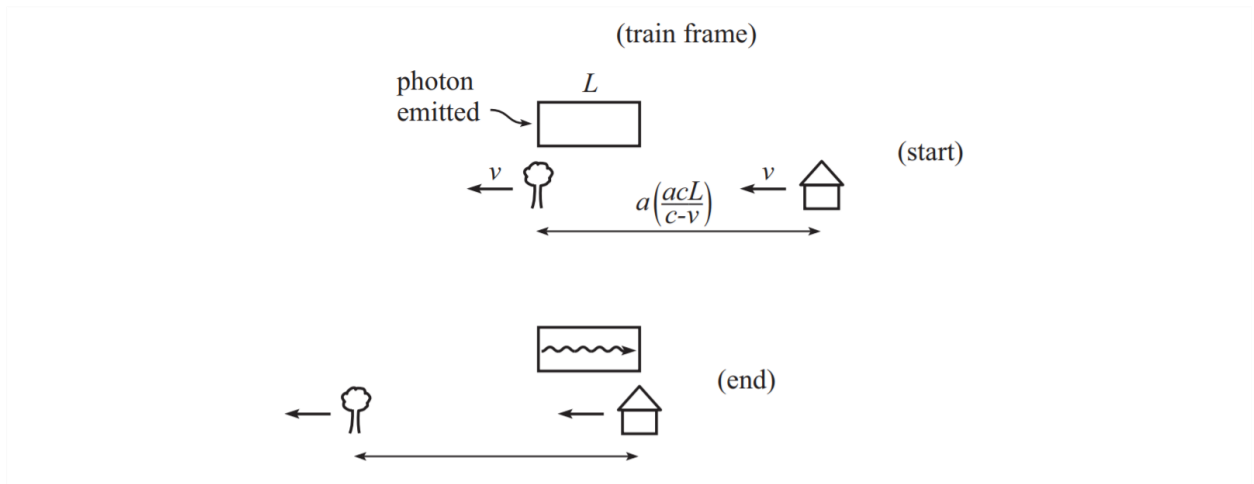
first postulate of relativity says that all inertial frames are equivalent, so the same $a_v L$ rule must apply to everyone. Consider the following setup. A train with proper length L moves with speed v . When the back of the train passes a tree, a photon is fired from the back toward the front. It arrives at the front when the front passes a house. What is the distance between the tree and the house (in the ground frame)?

Now look at things in the train frame. Using the tree-house proper distance you just found, write down the relation that expresses the fact that the house meets the front of the train at the same time the photon does. This will allow you to solve for a_v .

Solution

In the ground frame, the given rule tells us that the length of the train is $a_v L$. So the front of the train has a head start of $a_v L$ over the photon. The photon closes this gap at a relative speed of $c - v$, as measured in the ground frame. The time of the process is therefore $t = \frac{a_v L}{c - v}$. During this time, the photon travels a distance $c t = \frac{c a_v L}{c - v}$. This then is the tree-house distance in the ground frame.

Out[]=



Now consider the setup in the train frame as shown above. The photon is emitted when the tree coincides with the back of the train (a frame independent statement). The train has length L (its proper length), and the tree-house distance is $a_v (\frac{c a_v L}{c - v})$. This is true because our rule states that if an object is moving, the observed length equals a times the proper length, which we found above to be $\frac{c a_v L}{c - v}$ for the tree-house separation. The initial distance between the house and the front of the train is therefore $a \frac{c a_v L}{c - v} - L$ (subtracting off the train's length). The house covers this distance at speed v . The photon covers the length L of the train at speed c . Since the house and the photon arrive at the front of the train at the same time, we must therefore have

$$\frac{c a_v L}{c - v} - L = \frac{L}{c} \tag{13}$$

$$\frac{a_v^2}{1 - \frac{v}{c}} - 1 = \frac{v}{c} \tag{14}$$

$$a_v^2 = 1 - \frac{v^2}{c^2} \tag{15}$$

$$a_v = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma} \tag{16}$$

as desired. □

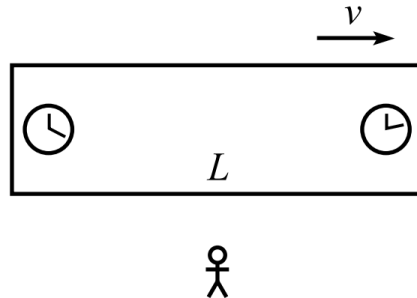
The Head Start Effect within the Lorentz Transformation

Example

Recall two facts about special relativity:

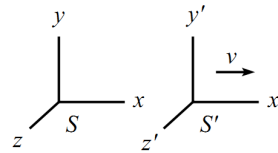
- **Head Start Effect:** If two synchronized clocks are at rest on a moving train (proper length L moving at speed v), an observer on the ground will see the rear clock ahead by $\frac{Lv}{c^2}$

Out[]=



- **Lorentz Equations:** Two events occurring a distance Δx and time Δt apart in frame S will occur a distance $\Delta x'$ apart and time $\Delta t'$ in a frame S' moving with speed v relative to S in the $+\hat{x}$ direction (as shown below)

Out[]=



$$\begin{aligned} \Delta x &= \gamma (\Delta x' + v \Delta t') & \Delta x' &= \gamma (\Delta x - v \Delta t) \\ \Delta t &= \gamma (\Delta t' + \frac{v}{c^2} \Delta x') & \Delta t' &= \gamma (\Delta t - \frac{v}{c^2} \Delta x) \end{aligned} \tag{17}$$

Show that the Lorentz equations contain the Head Start Effect. In other words, under the conditions implied by the Head Start Effect, pull out the time difference $\frac{Lv}{c^2}$ between the clock readings in the reference frame of the ground. Be careful about both the magnitude and sign of the effect (with the rear clock showing a higher time).

Solution

Denote the ground frame by S and the train frame by S' . We will define Event 1 as the person on the ground reading the front clock and Event 2 as the person on ground reading the rear clock. In the Head Start Effect, both Events happen simultaneously ($\Delta t = 0$), and the person reads the rear clock as showing a time $\frac{Lv}{c^2}$ larger than the front clock. How can we incorporate this information?

First, note that everyone must agree on the times that the front clock shows during Event 1 and the time the rear clock shows during Event 2 (for example, we could have set up the experiment so that two photons hits the two clocks when the person in the ground frame reads them, and the time on a clock when a photon hits it is frame independent). We know that the two clocks are synchronized on the train frame, which means that in that frame Event 2 must happen a time $\frac{Lv}{c^2}$ later than event 1, so we must have $\Delta t' = \frac{Lv}{c^2} = -\frac{\Delta x' v}{c^2}$ where we have used the fact that in the train's frame, the two events happen a distance $\Delta x' = -L$ apart (if you are confused by the negative sign, note that Event 2 happens at the back of the train (at a more negative \hat{x} value) at a later time).

The Lorentz equations do indeed give this result, since substituting $\Delta x = \gamma (\Delta x' + v \Delta t')$ and $\Delta t = 0$ into $\Delta t' = \gamma (\Delta t - \frac{v}{c^2} \Delta x)$ yields

$$\begin{aligned} \Delta t' &= \gamma (\Delta t - \frac{v}{c^2} \Delta x) \\ &= -\gamma^2 \frac{v}{c^2} (\Delta x' + v \Delta t') \end{aligned} \tag{18}$$

Solving for $\Delta t'$ yields

$$\Delta t' \left(1 + \frac{v^2}{c^2} \right) = -\gamma^2 \frac{v}{c^2} \Delta x' \tag{19}$$

$$\Delta t' \frac{1}{1 - \frac{v^2}{c^2}} = -\gamma^2 \frac{v}{c^2} \Delta x' \tag{20}$$

$$\Delta t' = -\frac{v}{c^2} \Delta x' \tag{21}$$

as desired. \square

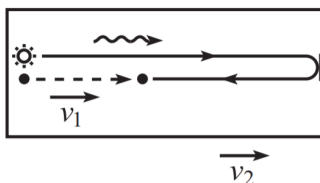
Insane Track

Velocity Addition from Scratch

Example

A ball moves at speed v_1 with respect to a train. The train moves at speed v_2 with respect to the ground. What is the speed of the ball with respect to the ground? Solve this problem (that is, derive the velocity-addition formula, $V = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$) in the following way. (Don't use time dilation, length contraction, etc. Use only the two postulates of relativity.)

Out[]=



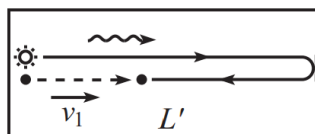
Let the ball be thrown from the back of the train. At the same instant, a photon is released next to it. The photon heads to the front of the train, bounces off a mirror, heads back, and eventually runs into the ball. In both the train frame and the ground frame, calculate the fraction of the way along the train where the meeting occurs, and then equate these fractions.

Solution

We will use the fact that the meeting of the photon and the ball occurs at the same fraction of the way along the train, independent of the frame. This is true because, although distances may change depending on the frame, fractions remain the same, since length contraction doesn't depend on position. We'll compute the desired fraction in the train frame S' and then in the ground frame S .

Out[]=

(train frame, S')



Train frame: Let the train have length L' in the train frame S' shown above. Let's first find the time at which the photon meets the ball. The sum of the distances traveled by the ball and the photon, which is $v_1 t' + c t'$, must equal twice the length of the train, which is $2L'$. The time of the meeting is therefore

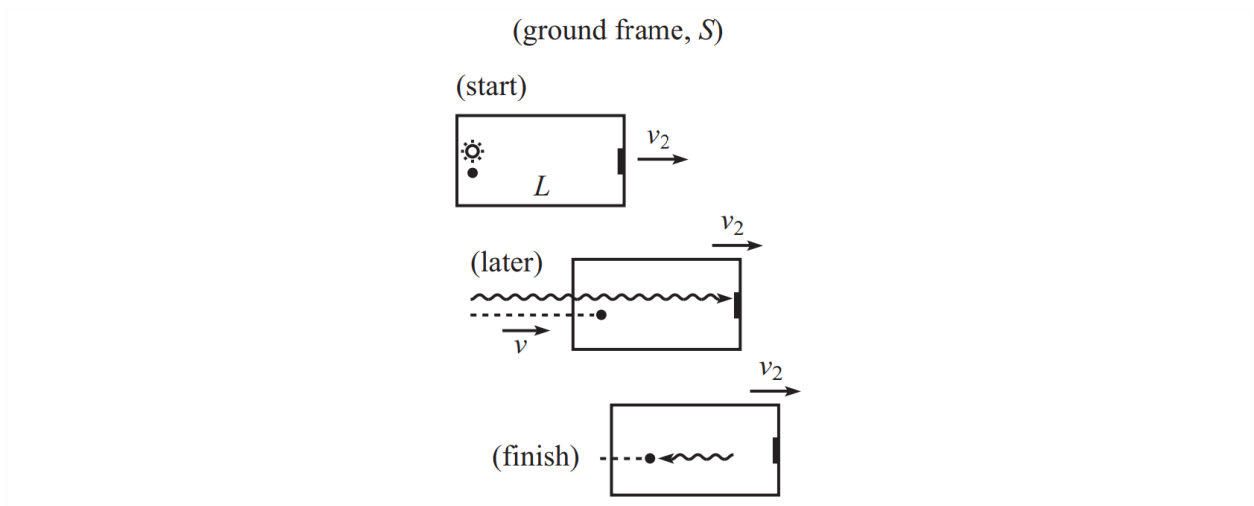
$$t' = \frac{2L'}{c + v_1} \tag{22}$$

The distance the ball has traveled is then $d' = v_1 t' = \frac{2v_1 L'}{c+v_1}$, so the desired fraction F' is

$$F' = \frac{d'}{L'} = \frac{2v_1}{c+v_1} \tag{23}$$

Ground frame: Let the speed of the ball with respect to the ground be v , and let the train have length L in the ground frame. (L equals $\frac{L'}{\gamma}$, but we're not going to use this.)

Out[]=



Again, let's first find the time at which the photon meets the ball. The photon takes a time $\frac{L}{c - v_2}$ to reach the mirror, because the initial gap of L is closed at a rate $c - v_2$ in the ground frame. At this time, the photon has traveled a distance $\frac{cL}{c - v_2}$. From the figure, we see that we can use the same reasoning we used in the train frame, but with the sum of the distances traveled by the ball and the photon, which is $v t + c t$, now equal to $\frac{2cL}{c - v_2}$. The time of the meeting in the ground frame is therefore

$$t = \frac{\left(\frac{2cL}{c - v_2}\right)}{c + v} \tag{24}$$

The relative speed of the ball and the back of the train (as viewed in the ground frame) is $v - v_2$. This is the rate at which the gap between them is increasing. So the distance between the ball and the back of the train at the time of the meeting is $d = (v - v_2) t = (v - v_2) \frac{2cL}{(c - v_2)(c + v)}$. The desired fraction F is therefore

$$F = \frac{d}{L} = \frac{2(v - v_2)c}{(c - v_2)(c + v)} \tag{25}$$

We can now equate the above expressions for F' and F , which yields

$$\frac{\frac{v_1}{c}}{1 + \frac{v_1}{c}} = \frac{\frac{v - v_2}{c}}{\left(1 - \frac{v_2}{c}\right)\left(1 + \frac{v}{c}\right)} \tag{26}$$

Solving for v in terms of v_1 and v_2 gives, after some algebra,

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \tag{27}$$

as desired. \square

Synchronizing Clocks

This problem demonstrates that if we take two synchronized clocks and move them arbitrarily slowly, they will stay synchronized.

Example

Two synchronized clocks, A and B , are at rest in a given frame, a distance L apart. A third clock, C , is initially

located right next to A . All three clocks have initial readings at zero, and then C is moved very slowly (with speed $v \ll c$) from A to B . Show that its final reading can be made arbitrarily close to B 's by making v sufficiently small. (The Taylor series $(1 - \epsilon)^{1/2} \approx 1 - \frac{\epsilon}{2}$ will come in handy.)

Solution

In the frame of A and B , it takes a time of $\frac{L}{v}$ for C to travel from A to B . During this time, C runs slow by a factor γ , so only $\frac{L}{\gamma v}$ elapses on C during the journey. Therefore, when C reaches B , the reading on C is $\frac{L}{\gamma v}$, and the reading on B is $\frac{L}{v}$. If v is small (more precisely, if $v \ll c$), we can use $(1 - \epsilon)^{1/2} \approx 1 - \frac{\epsilon}{2}$ to approximate C 's reading as

$$\frac{L}{\gamma v} = \frac{L}{v} \left(1 - \frac{v^2}{c^2}\right)^{1/2} \approx \frac{L}{v} \left(1 - \frac{v^2}{2c^2}\right) = \frac{L}{v} - \frac{Lv}{2c^2} \quad (28)$$

The difference between C 's and B 's readings is therefore $\frac{Lv}{2c^2}$. This goes to zero as $v \rightarrow 0$, as desired. We see that even though the total time $\frac{L}{v}$ in A 's and B 's frame goes to infinity as $v \rightarrow 0$, the $\frac{Lv}{2c^2}$ difference between C 's and B 's readings goes to zero. This is due to the fact that $\frac{L}{v}$ has only one power of v in the denominator, whereas the γ factor depends quadratically on v . If the γ factor were instead equal to $\frac{1}{(1 - \frac{v}{c})^{1/2}}$, then the difference in the readings would take on the fixed value of $\frac{L}{2c}$ in the $v \rightarrow 0$ limit. \square

Time Dilation and $\frac{Lv}{c^2}$

Example

A person walks very slowly at speed u from the back of a train of proper length L to the front. As shown in the previous problem, the time-dilation effect in the train frame can be made arbitrarily small by picking u to be sufficiently small, because the effect is second order in u , while the travel time is only first order in $\frac{1}{u}$. Therefore, if the person's watch agrees with a clock at the back of the train when he starts, then it also (essentially) agrees with a clock at the front when he finishes.

Now consider this setup in the ground frame, where the train moves at speed v . The rear clock reads $\frac{Lv}{c^2}$ more than the front, so in view of the preceding paragraph, the time gained by the person's watch during the process must be $\frac{Lv}{c^2}$ less than the time gained by the front clock (because they agree in the end). Since we are assuming u is small, you may assume $u \ll c$. By working in the ground frame, explain why this is the case.

Finally, consider the implications for the following scenario: If you line up a collection of these train systems around the circumference of a circular rotating platform. Let person A be at rest on the platform at a point on the circumference, and let person B start at A and walk arbitrarily slowly around the circumference. When B returns to A , how will B 's clock compare to A 's?

Solution

The velocity-addition formula gives the person's speed in the ground frame as $\frac{u+v}{1 + \frac{uv}{c^2}}$, so in the ground frame the person must close the initial gap of $\frac{L}{\gamma v}$ that the front of the train had at a relative speed of $\frac{u+v}{1 + \frac{uv}{c^2}} - v$. The time in the ground frame is therefore

$$t_g = \frac{L/\gamma v}{\frac{u+v}{1 + \frac{uv}{c^2}} - v} = \frac{L(1 + \frac{uv}{c^2})}{u(1 - \frac{v^2}{c^2})^{1/2}} \quad (29)$$

Compared with this ground-frame time, the front clock on the train runs slow by the factor γ_v , and the person's watch runs slow by the γ factor associated with the speed $\frac{u+v}{1+\frac{uv}{c^2}}$, which you can show equals $\gamma_u \gamma_v (1 + \frac{uv}{c^2})$. The difference in the elapsed times on the front clock and the person's watch is therefore

$$\begin{aligned} \Delta T_{\text{front}} - \Delta T_{\text{person}} &= \frac{L(1+\frac{uv}{c^2})}{u(1-\frac{v^2}{c^2})^{1/2}} \left(\frac{1}{\gamma_v} - \frac{1}{\gamma_u \gamma_v (1+\frac{uv}{c^2})} \right) \\ &= \frac{L}{u} \left(1 + \frac{uv}{c^2} - \frac{1}{\gamma_u} \right) \\ &= \frac{Lv}{c^2} + \frac{L}{u} \left(1 - \frac{1}{\gamma_u} \right) \end{aligned} \tag{30}$$

The second term here is negligible for the following reason. For small u , we can use the Taylor series $(1 - \epsilon)^{1/2} \approx 1 - \frac{\epsilon}{2}$ to write $\frac{1}{\gamma_u} = \left(1 - \frac{u^2}{c^2}\right)^{1/2} \approx 1 - \frac{u^2}{2c^2}$. The $\frac{L}{u} \left(1 - \frac{1}{\gamma_u}\right)$ term then becomes $\frac{L}{u} \frac{u^2}{2c^2}$. Since u is assumed to be small (more precisely, $u \ll c$), this term is negligible. So Equation (30) becomes $\Delta T_{\text{front}} - \Delta T_{\text{person}} \approx \frac{Lv}{c^2}$. The front clock therefore gains essentially $\frac{Lv}{c^2}$ more time than the person's watch, as we wanted to show.

Since the front clock started $\frac{Lv}{c^2}$ behind the person's watch, we conclude that they end up showing the same time when the watch reaches the front, as we already knew from working in the train frame. The point here is that no matter how small u is, the result for $\Delta T_{\text{front}} - \Delta T_{\text{person}}$ is nonzero (namely $\frac{Lv}{c^2}$) because u appears at first order in the γ factor, $\gamma_u \gamma_v (1 + \frac{uv}{c^2})$, associated with $\frac{u+v}{1+\frac{uv}{c^2}}$, while it appears only at second order in γ_u . The difference between the γ factors is therefore first order in u , and this difference combines with the $\frac{1}{u}$ factor in the time to yield a nonzero result.

The result in Equation (30) holds perfectly well for non-small u too, so it implies that the final readings on the front clock and the person's watch differ by $\frac{L}{u} \left(1 - \frac{1}{\gamma_u}\right)$, for any u . In retrospect, this is clear from the train-frame calculation which gives the difference as $\frac{L}{u} \left(1 - \frac{1}{\gamma_u}\right)$, due to the time dilation of the watch.

Regarding the final question, B 's clock will read less than A 's, because the above reasoning shows that an inertial observer sees B 's clock running slower than A 's. This result, that you can walk arbitrarily slowly in a particular reference frame and have your clock lose synchronization with other clocks, is a consequence of the fact that in some accelerating reference frames it is impossible to produce a consistent method (that is, one without a discontinuity) of clock synchronization. See this amusing article by [Cranor et al. \(2000\)](#) for more details. \square